

Generalization of One of Lie's Theorems

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Abstract

We prove a generalization of one of Lie's Theorems in the context of Lie-like algebras^{2-nd}.

The theory of Lie algebras has many applications in mathematics and physics. One possible way of generalizing the theory of Lie algebras is to develop the theory of Lie-like algebras^{2-nd} algebras, where the notion of a Lie-like algebras^{2-nd} algebra was introduced in [4]. One of Lie's Theorems claims that the only irreducible representations of a solvable Lie algebra over an algebraically closed field \mathbf{k} of characteristic 0 have dimension 1. We call this theorem Lie's Theorem for convenience in this paper. Since Lie's Theorem is one of the fundamental results in the theory of Lie algebras, finding the counterpart of Lie's Theorem in the context of Lie-like algebras^{2-nd} is of importance to develop the theory of Lie-like algebras^{2-nd}. The purpose of this paper is to prove a generalization of Lie's Theorem in the context of Lie-like algebras^{2-nd}.

In this paper, all vector spaces are vector spaces over an algebraically closed field \mathbf{k} of characteristic 0. The generalization of Lie's Theorem in this paper claims that a finite dimensional nonzero ordinary module over a finite dimensional solvable Lie-like algebras^{2-nd} always contains a one-dimensional ordinary submodule. Since a finite dimensional nonzero module over a finite dimensional solvable Lie superalgebra does not always contain a one-dimensional submodule ([1]), Lie-like algebras^{2-nd} are closer to Lie algebras than Lie superalgebras. Hence, using Lie-like algebras^{2-nd} seems to be more suitable than using Lie superalgebras in order to extend Lie theory.

After discussing the basic properties of modules over Lie-like algebras^{2-nd} in Section 1, we use Section 2 to give the proof of the generalization of Lie's Theorem.

1 Modules over Lie-like Algebras^{2-nd}

We begin this section with the definitions of a Lie-like algebra^{2-nd} L and an ordinary L -module([4]).

Definition 1.1 *Let S be a nonempty set. A vector space L is called a **Lie-like algebra**^{2-nd} induced by the set S if there exist a family of binary operations*

$$\left\{ \langle \cdot, \cdot \rangle_k \mid \langle \cdot, \cdot \rangle_k : L \times L \rightarrow L \text{ is a binary map and } k \in S \right\}$$

*such that both the **Jacobi-like identity**^{2-nd}*

$$\langle \langle x, y \rangle_k, z \rangle_h = \langle x, \langle y, z \rangle_h \rangle_k + \langle \langle x, z \rangle_h, y \rangle_k \quad (1)$$

and the following identity

$$\langle \langle x, y \rangle_k, z \rangle_h = \langle \langle x, y \rangle_h, z \rangle_k \quad (2)$$

hold for $x, y, z \in L$ and $h, k \in S$

A Lie-like algebra^{2-nd} L induced by the set S is also denoted by $(L, \langle \cdot, \cdot \rangle_{k \in S})$. The notion of a Leibniz algebra was introduced in [5]. Clearly, if L is a Lie-like algebra^{2-nd} induced by a set S , then L is a Leibniz algebra with respect to the binary operation $\langle \cdot, \cdot \rangle_k$ for each $k \in S$. Thus, a Leibniz algebra is a Lie-like algebra^{2-nd} L induced by the set whose cardinal number is 1, and a Lie-like algebra^{2-nd} induced by a set S is a bundle of Leibniz algebras satisfying the Jacobi-like identity^{2-nd} and (2).

A Lie-like algebra^{2-nd} $(L, \langle \cdot, \cdot \rangle_{k \in S})$ is said to be **non-trivial** if there do not exist a binary operation $\langle \cdot, \cdot \rangle : L \times L \rightarrow L$ and a map $\phi : S \rightarrow \mathbf{k}$ such that

$$\langle x, y \rangle_k = \phi(k) \langle x, y \rangle \quad \text{for } x, y \in L \text{ and } k \in S. \quad (3)$$

A subspace I of a Lie-like algebra^{2-nd} $(L, \langle \cdot, \cdot \rangle_{k \in S})$ is called an **ideal** of L if $\langle I, L \rangle_s \subseteq I$ and $\langle L, I \rangle_s \subseteq I$ for all $s \in S$.

Definition 1.2 *Let L be a Lie-like algebra^{2-nd} induced by a set S . A vector space V is called an **ordinary module** over L (or an **ordinary L -module**) if there exist a family of linear maps*

$$\left\{ f_k, g_k \mid \begin{array}{l} f_k : x \mapsto f_k(x) \text{ and } g_k : x \mapsto g_k(x) \text{ are linear} \\ \text{maps from } L \text{ to } \text{End}(V) \text{ for } x \in L \text{ and } k \in S \end{array} \right\}$$

such that

$$f_h(\langle x, y \rangle_k) = [f_h(x), f_k(y)], \quad (4)$$

$$g_h(\langle x, y \rangle_k) = [g_h(x), f_k(y)], \quad (5)$$

$$g_k(x)g_h(y) = g_h(x)f_k(y) = g_k(x)f_h(y), \quad (6)$$

and

$$f_k(x)f_h(y) = f_h(x)f_k(y), \quad f_k(x)g_h(y) = f_h(x)g_k(y), \quad (7)$$

where $x, y \in L$, and $[f_h(x), f_k(y)] := f_h(x)f_k(y) - f_k(y)f_h(x)$ is the ordinary bracket. An ordinary L -module V is also denoted by $(V, \{f_k\}, \{g_k\}_{k \in S})$, and f_k and g_k are called the **right linear map indexed by k** and the **left linear map indexed by k** of V , respectively.

The notion of a module over a Leibniz algebra, which was introduced in [6], is a special case of the notion of an ordinary module over a Lie-like algebra^{2-nd}. Clearly, an ordinary module over a Lie-like algebra^{2-nd} is a bundle of modules over Leibniz algebras.

If $(V, \{f_k\}, \{g_k\}_{k \in S})$ is an ordinary module over a Lie-like algebra^{2-nd} L , then

$$f_h(\langle x, y \rangle_k) = f_k(\langle x, y \rangle_h) \quad \text{for } x, y \in L \text{ and } h, k \in S \quad (8)$$

and

$$g_h(\langle x, y \rangle_k) = g_k(\langle x, y \rangle_h) \quad \text{for } x, y \in L \text{ and } h, k \in S. \quad (9)$$

If $(L, \langle, \rangle_{k \in S})$ is a Lie-like algebra^{2-nd}, then $(L, -r_k, \ell_k)$ is an ordinary module over the Lie-like algebra^{2-nd} L , where r_k and ℓ_k are the right multiplication and the left multiplication of L , respectively; that is,

$$r_k(x)(a) := \langle a, x \rangle_k \quad \text{and} \quad \ell_k(x)(a) := \langle x, a \rangle_k \quad \text{for } x, a \in L \text{ and } k \in S. \quad (10)$$

This ordinary L -module $(L, -r_k, \ell_k)$ is called the **adjoint module** over the Lie-like algebra^{2-nd} L .

Let $(V, \{f_k\}, \{g_k\}_{k \in S})$ be an ordinary module over a Lie-like algebra^{2-nd} $(L, \langle, \rangle_{k \in S})$. A subspace U of V is called an **ordinary submodule** of V if

$$f_k(x)(U) \subseteq U \quad \text{and} \quad g_k(x)(U) \subseteq U \quad \text{for } x \in L \text{ and } k \in S.$$

Following [4], the subspace

$$V_{2-nd}^{ann,+} := \sum_{\substack{x \in L, v \in V \\ h, k \in S}} \mathbf{k}(g_h(x) - f_k(x))(v) \quad (11)$$

is called the **plus annihilator** of V , which is an ordinary submodule of V and plays an important role in the proof of the generalization of Lie's Theorem.

A Lie-like algebra^{2-nd} $(L, \langle, \rangle_{k \in S})$ is said to be **solvable** if there exists a positive integer n such that $\mathcal{D}^n \mathcal{L} = 0$, where $\mathcal{D}^n \mathcal{L}$ is defined inductively as follows:

$$\mathcal{D}^1 \mathcal{L} := \mathcal{L}, \quad \mathcal{D}^{n+1} \mathcal{L} = \sum_{k \in S} \langle \mathcal{D}^n \mathcal{L}, \mathcal{D}^n \mathcal{L} \rangle_k \quad \forall n \geq 1. \quad (12)$$

2 The Proof of the Main Theorem

In order to present the proof of the generalization of Lie's Theorem in a clear way, we divide it into a few parts. First, we have

Proposition 2.1 *Let V be a finite dimensional vector space over \mathbf{k} , and let \mathcal{G} be a subspace of the general linear Lie algebra $gl(V)$. If \mathcal{A} is a subspace of $gl(V)$ and*

$$\mathcal{G} \subseteq \mathcal{N}_{gl(V)}(\mathcal{A}) := \{ X \in gl(V) \mid [\mathcal{A}, X] \subseteq \mathcal{A} \},$$

then

$$U := \{ u \in V \mid A(u) = \phi(A)u \text{ for all } A \in \mathcal{A} \}$$

is an invariant subspace of \mathcal{G} ; that is, $\mathcal{G}(U) \subseteq U$, where $\phi : \mathcal{A} \rightarrow \mathbf{k}$ is a linear functional.

Proof The proof of Proposition 2.1 is the same as the proof of Lemma 2 on page 24 in [7]. □

Next, we have

Proposition 2.2 *Let $L = \mathcal{A} \oplus \mathbf{k}x$ (direct sum of vector spaces) be a Lie-like algebra^{2-nd} over the field \mathbf{k} , where \mathcal{A} is an ideal of L and $\langle L, L \rangle_k \subseteq \mathcal{A}$ for $k \in S$. Let $(V, \{f_k\}, \{g_k\}_{k \in S})$ be a finite dimensional ordinary L -module such that*

$$f_k(A)(u_0) = \phi_k(A)u_0, \quad g_k(A)(u_0) = \psi_k(A)u_0 \quad \text{for } A \in \mathcal{A} \text{ and } k \in S, \quad (13)$$

where $u_0 \in V$, $\phi_k : \mathcal{A} \rightarrow \mathbf{k}$ and $\psi_k : \mathcal{A} \rightarrow \mathbf{k}$ are functionals. If $h \in S$ and $u_m := g_h(x)^m(u_0)$ for $m \in \mathbb{Z}_{\geq 0}$, then

$$f_k(A)(u_m) \equiv \phi_k(A)u_m \mod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^{m-1} \mathbf{k}u_i \right), \quad (14)$$

$$g_k(A)(u_m) \equiv \psi_k(A)u_m \mod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^{m-1} \mathbf{k}u_i \right) \quad (15)$$

and

$$f_k(x)(u_m) \equiv u_{m+1} \mod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^m \mathbf{k}u_i \right) \quad (16)$$

for all $A \in \mathcal{A}$ and $k \in S$.

Proof We use induction on m to prove (14), (15) and (16). If $m = 0$, then both (14) and (15) hold by (13), and (16) holds by (11).

We now prove that if (14), (15) and (16) hold for a fixed nonnegative integer n with $n \leq m$, then

$$f_k(A)(u_{m+1}) \equiv \phi_k(A)u_{m+1} \bmod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^m \mathbf{k}u_i \right), \quad (17)$$

$$g_k(A)(u_{m+1}) \equiv \psi_k(A)u_{m+1} \bmod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^m \mathbf{k}u_i \right) \quad (18)$$

and

$$f_k(x)(u_{m+1}) \equiv u_{m+2} \bmod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^{m+1} \mathbf{k}u_i \right) \quad (19)$$

for all $A \in \mathcal{A}$ and $k \in S$.

Since $V_{2-nd}^{ann,+}$ is a submodule of V , we have

$$f_k(z)(V_{2-nd}^{ann,+}) \subseteq V_{2-nd}^{ann,+} \quad \text{for } z \in L \text{ and } k \in S. \quad (20)$$

Using (5), we have

$$\begin{aligned} f_k(A)(u_{m+1}) &= f_k(A)g_h(x)(u_m) = \left(-[g_h(x), f_k(A)] + g_h(x)f_k(A) \right)(u_m) \\ &= -g_h(\langle x, A \rangle_k)(u_m) + g_h(x)f_k(A)(u_m). \end{aligned} \quad (21)$$

Since \mathcal{A} is an ideal of L , $\langle x, A \rangle_k \in \mathcal{A}$. Hence, we get

$$g_h(\langle x, A \rangle_k)(u_m) \stackrel{(15)}{\equiv} \psi_h(\langle x, A \rangle_k)u_m \bmod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^{m-1} \mathbf{k}u_i \right)$$

or

$$g_h(\langle x, A \rangle_k)(u_m) \equiv 0 \bmod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^m \mathbf{k}u_i \right). \quad (22)$$

Using the fact that $g_h(x)(V_{2-nd}^{ann,+}) \subseteq V_{2-nd}^{ann,+}$, we get

$$\begin{aligned} &g_h(x)(f_k(A)(u_m)) \\ \stackrel{(14)}{\equiv} &g_h(x)(\phi_k(A)u_m) \bmod \left(g_h(x)(V_{2-nd}^{ann,+}) + \sum_{i=0}^{m-1} \mathbf{k}g_h(x)(u_i) \right) \\ \equiv &\phi_k(A)g_h(x)(u_m) \bmod \left(V_{2-nd}^{ann,+} + \sum_{i=1}^m \mathbf{k}u_i \right) \\ \equiv &\phi_k(A)u_{m+1} \bmod \left(V_{2-nd}^{ann,+} + \sum_{i=1}^m \mathbf{k}u_i \right). \end{aligned} \quad (23)$$

It follows from (21), (22) and (23) that (17) holds.

Similarly, one can prove that (18) holds.

Finally, using the fact that $\langle x, x \rangle_k \in \mathcal{A}$, we have

$$\begin{aligned}
f_k(x)(u_{m+1}) &= f_k(x)g_h(x)(u_m) = \left(-[g_h(x), f_k(x)] + g_h(x)f_k(x) \right)(u_m) \\
&\stackrel{(5)}{=} -g_h(\langle x, x \rangle_k)(u_m) + g_h(x)f_k(x)(u_m) \\
&\stackrel{(15)}{=} 0 + g_h(x)f_k(x)(u_m) \bmod \left(V_{2-nd}^{ann,+} + \sum_{i=0}^m \mathbf{k}u_i \right) \\
&\stackrel{(16)}{=} g_h(x)(u_{m+1}) \bmod \left(g_h(x)(V_{2-nd}^{ann,+}) + \sum_{i=0}^m \mathbf{k}g_h(x)(u_i) \right) \\
&\equiv u_{m+2} \bmod \left(\sum_{i=1}^{m+1} \mathbf{k}u_i \right),
\end{aligned}$$

which proves that (19) holds. \square

Using Proposition 2.2, we now prove the following

Proposition 2.3 *Let $L = \mathcal{A} \oplus \mathbf{k}x$ (direct sum of vector spaces) be a Lie-like algebra^{2-nd} over the field \mathbf{k} , where \mathcal{A} is an ideal of L and $\langle L, L \rangle_k \subseteq \mathcal{A}$ for $k \in S$. Let $\phi_k : \mathcal{A} \rightarrow \mathbf{k}$ and $\psi_k : \mathcal{A} \rightarrow \mathbf{k}$ be functionals for $k \in S$. If $(V, \{f_k\}, \{g_k\}_{k \in S})$ is a finite dimensional ordinary L -module, then*

$$f_h(x)(U_\phi \cap U_\psi) \subseteq U_\phi \cap U_\psi \quad \text{for all } h \in S \quad (24)$$

and

$$g_h(x)(u) \subseteq U_\phi \quad \text{for all } h \in S \text{ and } u \in (U_\phi \cap U_\psi) \setminus V_{2-nd}^{ann,+}, \quad (25)$$

where

$$U_\phi := \bigcap_{k \in S} U_{\phi_k}, \quad U_\psi := \bigcap_{k \in S} U_{\psi_k},$$

$$U_{\phi_k} := \{ u \in V \mid f_k(A)(u) = \phi_k(A)u \text{ for } A \in \mathcal{A} \}$$

and

$$U_{\psi_k} := \{ u \in V \mid g_k(A)(u) = \psi_k(A)u \text{ for } A \in \mathcal{A} \}.$$

Proof Since \mathcal{A} is an ideal of L , we have

$$[f_h(x), f_k(A)] \stackrel{(4) \& (8)}{=} f_k(\langle x, A \rangle_h) \in f_k(\mathcal{A}) \Rightarrow f_h(x) \in \mathcal{N}_{g_\ell(V)}(f_k(\mathcal{A})) \quad (26)$$

and

$$[g_k(A), f_h(x)] \stackrel{(5)}{=} g_k(\langle A, x \rangle_h) \in g_k(\mathcal{A}) \Rightarrow f_h(x) \in \mathcal{N}_{g_\ell(V)}(g_k(\mathcal{A})). \quad (27)$$

By (26) and Proposition 2.1, $f_h(x)$ does not change U_{ϕ_k} , which implies that

$$f_k(A)\left(f_h(x)(u)\right) = \phi_k(A)f_h(x)(u) \quad \text{for } A \in \mathcal{A}, u \in U_{\phi_k} \text{ and } k, h \in S. \quad (28)$$

Similarly, by (27) and Proposition 2.1, $f_h(x)$ does not change U_{ψ_k} , which implies that

$$g_k(A)\left(f_h(x)(u)\right) = \psi_k(A)f_h(x)(u) \quad \text{for } A \in \mathcal{A}, u \in U_{\psi_k} \text{ and } k, h \in S. \quad (29)$$

Using (28) and (29), we get (24).

We now prove

$$f_k(A)\left(g_h(x)(u)\right) = \phi_k(A)g_h(x)(u) \quad (30)$$

for $A \in \mathcal{A}$, $k, h \in S$ and $u \in (U_\phi \cap U_\psi) \setminus V_{2-nd}^{ann,+}$.

Let h be an arbitrary element of S . We define $u_0 := u$ and $u_m := g_h(x)^m(u_0)$ for $m \in \mathbb{Z}_{\geq 0}$. Since V is finite dimensional and $u_m \in V$ for $m \in \mathbb{Z}_{\geq 0}$, there exists a positive integer p such that

$$W_h := V_{2-nd}^{ann,+} \oplus \mathbf{k}u_0 \oplus \cdots \oplus \mathbf{k}u_{p-1} \text{ and } u_m \in W \text{ for } m \in \mathbb{Z}_{\geq 0}. \quad (31)$$

By (14) and (20), we have

$$f_k(A)(W_h) \subseteq W_h \quad \text{for } A \in \mathcal{A} \text{ and } k \in S. \quad (32)$$

By (15) and (20), we have

$$g_k(A)(W_h) \subseteq W_h \quad \text{for } A \in \mathcal{A} \text{ and } k \in S. \quad (33)$$

Clearly, we have

$$g_h(x)(W_h) \subseteq W_h. \quad (34)$$

It follows that W_h is invariant under $f_k(\mathcal{A})$, $g_k(\mathcal{A})$ and $g_h(x)$. Combining a basis of $V_{2-nd}^{ann,+}$ and the linearly independent vectors u_0, u_1, \dots, u_{p-1} , we get a basis of W_h . With respect to this basis of W_h , the matrix of $g_h(\langle x, A \rangle_k)|_{W_h} = [g_h(x), f_k(A)]|_{W_h} \in \text{End}(W_h)$ has the form

$$\left(\begin{array}{c|cccc} M_{\dim V_{2-nd}^{ann,+} \times \dim V_{2-nd}^{ann,+}} & \text{a } (\dim V_{2-nd}^{ann,+}) \times p \text{ matrix} & & & \\ \hline & \psi_h(\langle x, A \rangle_k) & * & \cdots & * \\ & 0 & \psi_h(\langle x, A \rangle_k) & \cdots & * \\ & \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & \cdots & \psi_h(\langle x, A \rangle_k) \end{array} \right)$$

by (15), where $M_{\dim V_{2-nd}^{ann,+} \times \dim V_{2-nd}^{ann,+}}$ is the matrix of $[g_h(x), f_k(A)]|_{V_{2-nd}^{ann,+}}$ with respect to the basis of $V_{2-nd}^{ann,+}$. Hence, we get

$$\text{tr} M = \text{tr}([g_h(x), f_k(A)]|_{V_{2-nd}^{ann,+}}) = 0$$

and

$$0 = \text{tr}([g_h(x), f_k(A)]|W) = \text{tr}M + p\psi_h(\langle x, A \rangle_k) = p\psi_h(\langle x, A \rangle_k),$$

which implies that

$$\psi_h(\langle x, A \rangle_k) = 0 \quad \text{for } A \in \mathcal{A} \text{ and } h, k \in S. \quad (35)$$

Using (35), we have

$$\begin{aligned} f_k(A)(g_h(x)(u)) &= -([g_h(x), f_k(A)] + g_h(x)f_k(A))(u) \\ &= -g_h(\langle x, A \rangle_k)(u) + g_h(x)(f_k(A)(u)) \\ &= -\psi_h(\langle x, A \rangle_k)(u) + g_h(x)(\phi_k(A)(u)) = \phi_k(A)g_h(x)(u), \end{aligned}$$

which proves that (30) also holds. By (30), (25) is true. \square

The following counterpart of Lie's Theorem claims that if L is a finite dimensional solvable Lie-like algebra^{2-nd} over an algebraically closed field of characteristic 0, then every finite dimensional non-zero ordinary L -module always contains an one dimensional ordinary submodule.

Proposition 2.4 (The Generalization of Lie's Theorem) *Let L be a finite dimensional solvable Lie-like algebra^{2-nd} over an algebraically closed field of characteristic 0. If $(V, \{f_k\}, \{g_k\}_{k \in S})$ is a finite dimensional ordinary L -module, then there exist linear functionals $\phi_k, \psi_k : L \rightarrow \mathbf{k}$ for $k \in S$ and a nonzero vector v in V such that*

$$f_k(z)(v) = \phi_k(z)v, \quad g_k(z)(v) = \psi_k(z)v \quad \text{for } z \in L \text{ and } k \in S. \quad (36)$$

Moreover, either $\psi_k = 0$ for all $k \in S$ or $\phi_k = \psi_k$ for all $k \in S$.

Proof We use induction on $n := \dim L$. If $n = 0$, then Proposition 2.4 is clear true.

Assume that Proposition 2.4 is true for all finite dimensional solvable Lie-like algebra^{2-nd} of dimensions smaller than n . Note that any subspace of L containing $\mathcal{D}^2\mathcal{L} = \sum_{k \in S} \langle L, L \rangle_k$ is an ideal of L . Since $n = \dim L \geq 1$ and L is solvable, $L \neq \mathcal{D}^2\mathcal{L}$. Therefore, we can find an ideal \mathcal{A} of L such that $\langle L, L \rangle_k \subseteq \mathcal{A}$ for $k \in S$ and $L = \mathcal{A} \oplus \mathbf{k}x$ for some $x \notin \mathcal{A}$.

Now \mathcal{A} is a solvable Lie-like algebra^{2-nd} of dimension smaller than n and $(V, \{f_k\}, \{g_k\}_{k \in S})$ is a finite dimensional \mathcal{A} -module. It follows from induction assumption that

$$U := \left\{ u \in V \left| \begin{array}{l} \text{there exist linear functionals } \phi'_k : \mathcal{A} \rightarrow \mathbf{k} \\ \text{and } \psi'_k : \mathcal{A} \rightarrow \mathbf{k} \text{ for } k \in S \text{ such that} \\ f_k(A)(u) = \phi'_k(A)u, g_k(A)(u) = \psi'_k(A)u \\ \text{for } A \in \mathcal{A} \text{ and } k \in S \end{array} \right. \right\} \neq 0,$$

Then we have either $U \cap V_{2-nd}^{ann,+} \neq 0$ or $U \cap V_{2-nd}^{ann,+} = 0$.

First, we assume that $U \cap V_{2-nd}^{ann,+} \neq 0$. By (20) and (24), we have

$$f_k(x) \left(U \cap V_{2-nd}^{ann,+} \right) \subseteq U \cap V_{2-nd}^{ann,+} \quad \text{for } k \in S.$$

Hence, $\{f_k(x) | (U \cap V_{2-nd}^{ann,+}) | k \in S\}$ is a set of commutative linear transformations of the vector space $U \cap V_{2-nd}^{ann,+}$ by (7). Thus, there exists $0 \neq v_0 \in U \cap V_{2-nd}^{ann,+}$ and $\lambda_k \in \mathbf{k}$ such that

$$f_k(x)(v_0) = \lambda_k v_0 \quad \text{for } k \in S,$$

which implies that

$$f_k(z)(v_0) = \phi_k(z)v_0 \quad \text{for } z \in L \text{ and } k \in S, \quad (37)$$

where the functional $\phi_k : L \rightarrow \mathbf{k}$ is defined by

$$\phi_k(z) = \begin{cases} \phi'_k(z) & \text{if } z \in \mathcal{A}, \\ \lambda_k & \text{if } z = x. \end{cases}$$

If $g_k(x)(v_0) = 0$ for $k \in S$, then

$$g_k(z)(v_0) = \psi_k(z)v_0 \quad \text{for } z \in L \text{ and } k \in S, \quad (38)$$

where the functional $\psi_k : L \rightarrow \mathbf{k}$ is defined by

$$\psi_k(z) = \begin{cases} \psi'_k(z) & \text{if } z \in \mathcal{A}, \\ 0 & \text{if } z \in \mathbf{k}x. \end{cases}$$

Hence, (36) holds for $v := v_0$ by (37) and (38).

If $g_{h_0}(x)(v_0) \neq 0$ for some $h_0 \in S$, then (36) holds for $v := g_{h_0}(x)(v_0)$ and $\phi_k = \psi_k = 0$ with $k \in S$. In fact, by (6) and (7), we have

$$f_k(z)g_{h_0}(x)(g_s(y) - f_t(y)) = 0 = g_k(z)g_{h_0}(x)(g_s(y) - f_t(y)) \quad (39)$$

for $z, y \in L$ and $k, s, t \in S$. Since $v_0 \in V_{2-nd}^{ann,+}$, v_0 is a linear combination of the vectors in the set $\{(g_s(y) - f_t(y))(w) | s, t \in S, y \in L \text{ and } w \in V\}$. It follows from this fact and (39) that

$$f_k(z)(g_{h_0}(x)(v_0)) = 0 = g_k(z)(g_{h_0}(x)(v_0)) \quad \text{for } z \in L \text{ and } k \in S,$$

which implies that (36) holds for $v := g_{h_0}(x)(v_0)$ and $\phi_k = \psi_k = 0$ with $k \in S$.

We now assume that $U \cap V_{2-nd}^{ann,+} = 0$, in which case, we prove (36) by cases. Using the same notations as the ones in Proposition 2.3, we have $U = U_\phi \cap U_\psi$.

Case 1: $f_{h_0}(x)(w) \neq g_{h_0}(x)(w)$ for some $w \in U$ and some $h_0 \in S$, in which case, let $\tilde{w} := (f_{h_0}(x) - g_{h_0}(x))(w)$. Then $\tilde{w} \neq 0$. For any $A \in \mathcal{A}$, we have

$$\begin{aligned}
f_k(A)(\tilde{w}) &= f_k(A)(f_{h_0}(x) - g_{h_0}(x))(w) \\
&= f_k(A)\left(f_{h_0}(x)(w)\right) - f_k(A)\left(g_{h_0}(x)(w)\right) \\
&\stackrel{(24)}{=} \phi'_k(A)f_{h_0}(x)(w) - f_k(A)\left(g_{h_0}(x)(w)\right) \\
&\stackrel{(25)}{=} \phi'_k(A)f_{h_0}(x)(w) - \phi'_k(A)g_{h_0}(x)(w) \\
&= \phi'_k(A)(f_{h_0}(x) - g_{h_0}(x))(w) = \phi'_k(A)\tilde{w}
\end{aligned}$$

or

$$f_k(A)(\tilde{w}) = \phi'_k(A)\tilde{w} \quad \text{for all } A \in \mathcal{A} \text{ and } k \in S. \quad (40)$$

By (6), we have

$$g_k(A)(\tilde{w}) = g_k(A)(f_{h_0}(x) - g_{h_0}(x))(w) = 0 \quad \text{for } A \in \mathcal{A} \text{ and } k \in S. \quad (41)$$

Let

$$\tilde{U} := \{ \tilde{u} \in V \mid f_k(A)(\tilde{u}) = \phi'_k(A)\tilde{u}, g_k(A)(\tilde{u}) = 0 \text{ for all } A \in \mathcal{A} \text{ and } k \in S \}.$$

It follows from (40) and (41) that $0 \neq \tilde{w} \in \tilde{U} \cap V_{2-nd}^{ann,+}$, in which case, we have proved that (36) holds.

Case 2: $f_h(x)(u) = g_h(x)(u)$ for all $h \in S$ and all $u \in U$, in which case, for $A \in \mathcal{A}$, we have

$$g_k(A)\left(g_h(x)(u)\right) \stackrel{(6)}{=} g_k(A)\left(f_h(x)(u)\right) \stackrel{(24)}{=} \psi_k(A)\left(f_h(x)(u)\right) = \psi_k(A)(g_h(x)(u))$$

or

$$g_h(x)(u) \in U_\psi \quad \text{for } u \in U = U_\phi \cap U_\psi \text{ and } h \in S. \quad (42)$$

By (25) and (42), $g_h(x)(U) \subseteq U_\phi \cap U_\psi = U$ for $h \in S$. This fact and (24) prove that U is invariant under both $f_k(x)$ and $g_k(x)$ for $k \in S$. In particular, the set $\{f_k(x)|U \mid k \in S\}$ consisting of commutative linear transformations has a common eigenvector in U . Hence, there exists $\lambda \in \mathbf{k}$ such that

$$U^\lambda := \{u \in U \mid f_k(x)(u) = \lambda_k u \text{ for } k \in S\} \neq 0.$$

Let $0 \neq u_0 \in U^\lambda$ and $u_m := g_h(x)^m(u_0)$ for $m \in \mathbb{Z}_{\geq 0}$, where h is a arbitrary fixed element of S . We now use induction on m to prove

$$f_k(x)(u_m) \equiv \lambda_k u_m \pmod{\left(\sum_{i=0}^{m-1} \mathbf{k} u_i\right)} \quad \text{for } m \in \mathbb{Z}_{\geq 0} \text{ and } k \in S. \quad (43)$$

Clearly, (43) holds for $m = 0$. Assume that (43) holds for m . Then we get

$$\begin{aligned}
f_k(x)(u_{m+1}) &= f_k(x)g_h(x)(u_m) = \left(-[g_h(x), f_k(x)] + g_h(x)f_k(x)\right)(u_m) \\
&= -g_h(\langle x, x \rangle_k)(u_m) + g_h(x)f_k(x)(u_m).
\end{aligned} \quad (44)$$

Since $u_m \in U$ for $m \in \mathbb{Z}_{\geq 0}$ and $\langle x, x \rangle_k \in \mathcal{A}$, we get

$$g_h(\langle x, x \rangle_k)(u_m) = \psi'_h(\langle x, x \rangle_k)u_m. \quad (45)$$

Using (43), we get

$$\begin{aligned} g_h(x)f_k(x)(u_m) &\equiv g_h(x)(\lambda_k u_m) \bmod \left(\sum_{i=0}^{m-1} \mathbf{k}g_h(x)(u_i) \right) \\ &\equiv \lambda_k g_h(x)(u_m) \bmod \left(\sum_{i=1}^m \mathbf{k}u_i \right) \\ &\equiv \lambda_k u_{m+1} \bmod \left(\sum_{i=1}^m \mathbf{k}u_i \right). \end{aligned} \quad (46)$$

By (44), (45) and (46), we have

$$\begin{aligned} f_k(x)(u_{m+1}) &\equiv -\psi'_h(\langle x, x \rangle_k)u_m + \lambda_k u_{m+1} \bmod \left(\sum_{i=1}^m \mathbf{k}u_i \right) \\ &\equiv \lambda_k u_{m+1} \bmod \left(\sum_{i=1}^m \mathbf{k}u_i \right), \end{aligned}$$

which proves that (43) also holds for $m+1$.

Since V is finite dimensional, there exists a positive integer p such that

$$W_h := \mathbf{k}u_0 \oplus \cdots \oplus \mathbf{k}u_{p-1} \quad \text{and} \quad g_h(x)(W) \subseteq W. \quad (47)$$

By (43) and (47), W_h is invariant under both $f_k(x)$ for all $k \in S$ and $g_h(x)$. Since

$$[f_k(x), g_h(x)](u_m) \stackrel{(5)}{=} -g_h(\langle x, x \rangle_k)(u_m) \stackrel{(45)}{=} -\psi'_h(\langle x, x \rangle_k)u_m,$$

the matrix of $[f_k(x), g_h(x)]|_{W_h}$ in the basis u_0, \dots, u_{p-1} of W is the diagonal matrix:

$$[f_k(x), g_h(x)]|_{W_h} = \begin{pmatrix} -\psi'_h(\langle x, x \rangle_k) & 0 & \cdots & 0 \\ 0 & -\psi'_h(\langle x, x \rangle_k) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\psi'_h(\langle x, x \rangle_k) \end{pmatrix}_{p \times p}$$

Hence, $\text{tr}([f_k(x), g_h(x)]|_W) = -p\psi'_h(\langle x, x \rangle_k)$, which implies that $\psi'_h(\langle x, x \rangle_k) = 0$. Now for any $u \in U^\lambda$, we have

$$\begin{aligned} f_k(x)(g_h(x)(u)) &= ([f_k(x), g_h(x)] + g_h(x)f_k(x))(u) \\ &= -g_h(\langle x, x \rangle_k)(u) + g_h(x)f_k(x)(u) \\ &= -\psi'_h(\langle x, x \rangle_k)u + g_h(x)f_k(x)(u) = 0 + \lambda_k g_h(x)(u), \end{aligned}$$

which implies that $g_h(x)(U^\lambda) \subseteq U^\lambda$. This proves that U^λ is an invariant subspace of $g_h(x)$ for all $h \in S$. Since the set $\{g_k(x)|U^\lambda \mid k \in S\}$ of commutative linear transformations is commutative, there exists $0 \neq u' \in U^\lambda$ such that $g_k(x)(u') = \mu_k u'$ for $k \in S$ and $\mu_k \in \mathbf{k}$.

Summarizing what we know about $u' \in U^\lambda \subseteq U$, we have

$$f_k(A)(u') = \phi'_k(A)u', \quad g_k(A)(u') = \psi'_k(A)u' \quad \text{for } A \in \mathbf{A} \text{ and } k \in S$$

and

$$f_k(x)(u') = \lambda_k u', \quad g_k(x)(u') = \mu_k u' \quad \text{for } A \in \mathbf{A} \text{ and } k \in S.$$

Hence, if we define linear functionals $\phi_k, \psi_k : L \rightarrow \mathbf{k}$ by

$$\begin{aligned} \phi_k|_{\mathcal{A}} &:= \phi'_k, & \phi_k(x) &:= \lambda_k, \\ \psi_k|_{\mathcal{A}} &:= \psi'_k, & \psi_k(x) &:= \mu_k \end{aligned}$$

for $k \in S$, then

$$f_k(z)(u') = \phi_k(z)u', \quad g_k(z)(u') = \psi_k(z)u', \quad \text{for } z \in L \text{ and } k \in S, \quad (48)$$

which implies that (36) holds for $v := u'$.

Finally, if $\psi_h \neq 0$ for some $h \in S$, then there exists $y \in L$ such that $\psi_h(y) \neq 0$. Let v be a nonzero vector satisfying (36). For all $z \in L$ and all $k \in S$, we have

$$g_h(y)f_k(z)(v) = g_h(y)(\phi_k(z)v) = \phi_k(z)\phi_h(y)v \quad (49)$$

and

$$g_h(y)g_k(z)(v) = g_h(y)(\psi_k(z)v) = \psi_k(z)\psi_h(y)v \quad (50)$$

by (36). It follows from (49) and (50) that

$$(\phi_k(z) - \psi_k(z))\psi_h(y)(v) = (g_h(y)f_k(z) - g_h(y)g_k(z))(v) \stackrel{(6)}{=} 0(v) = 0. \quad (51)$$

Since $\psi_h(y) \neq 0$ and $v \neq 0$, (51) implies that $\phi_k(z) = \psi_k(z)$ for all $z \in L$ and all $k \in S$. It follows from that $\phi_k = \psi_k$ for all $k \in S$. \square

As a corollary of Proposition 2.4, we have the following

Proposition 2.5 *Let L be a finite dimensional solvable Leibniz algebra over an algebraically closed field of characteristic 0. If (V, f, g) is a finite dimensional nonzero L -module, then there exist two linear functionals $\phi, \psi : L \rightarrow \mathbf{k}$ and a nonzero vector v in V such that*

$$f(z)(v) = \phi(z)v, \quad g(z)(v) = \psi(z)v \quad \text{for } z \in L.$$

Moreover, either $(\phi, \psi) = (\phi, \phi)$ or $(\phi, \psi) = (\phi, 0)$.

Proof Since a solvable Leibniz algebra is a solvable Lie-like algebra^{2-nd}, Proposition 2.5 follows from Proposition 2.4 directly. □

Proposition 2.5, which was announced in [2], has been used to classify the simple Leibniz algebras with Lie factor sl_2 in [3].

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